INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF OPTICS A: PURE AND APPLIED OPTICS

J. Opt. A: Pure Appl. Opt. **5** (2003) 43–46 PII: S1464-4258(03)53038-0

Light reflection and transmission by non-absorbing turbid slabs: simple approximations

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Received 2 September 2002, in final form 14 October 2002
Published 9 December 2002
Online at stacks.iop.org/JOptA/5/43

Abstract
A simple approximation is developed for the total reflectance and transmittance of non-absorbing turbid plane-parallel layers. It is based on the energy conservation law and asymptotic equations for optically thick turbid slabs. The accuracy of the approximation is better than 1.5% at optical thickness larger than unity. This is comparable with the accuracy of many numerical schemes for the solution of the radiative transfer equation.

Keywords: Radiative transfer, light scattering, turbid media

1. Introduction

Radiative transfer in turbid media plays an important role in various branches of modern optics, including bio-optics and liquid crystals optics, as well as oceanic and atmospheric research [1]. Diffuse transmittance t and reflectance r (or spherical albedo) are routinely measured to extract information on the microstructure of the medium from light scattering experiments [2, 3]. Thus, it is of importance to have simple and yet accurate equations, which can be used for their calculation. This allows us to avoid numerical calculations, involving the solution of the integro-differential radiative transfer equation [1].

It has been found that r and t depend mostly on the optical thickness τ and the asymmetry parameter g of the turbid medium (both for large [1] and small [4] optical thickness τ). We exploit these findings together with the energy conservation law to derive simple approximations for the diffuse transmittance and reflectance of turbid slabs.

The study is limited to the case of homogeneous non-absorbing plane-parallel turbid media.

2. Approximations

The diffuse reflectance r and diffuse transmittance t are defined by the following equations [1]:

$$r = \int_0^{\pi/2} \mathrm{d}\vartheta_0 \sin(2\vartheta_0)\Re(\vartheta_0),$$
$$\Re(\vartheta_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\vartheta \sin(2\vartheta) R(\vartheta_0, \vartheta, \varphi), \quad (1)$$

$$t = \int_0^{\pi/2} \mathrm{d}\vartheta_0 \sin(2\vartheta_0) T(\vartheta_0),$$
$$T(\vartheta_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\vartheta \sin(2\vartheta) T(\vartheta_0, \vartheta, \varphi), \quad (2)$$

where ϑ_0 is the light incidence angle, ϑ is the observation angle, $\varphi = \varphi_2 - \varphi_1$, φ_1 is the azimuth angle of the incident light beam, φ_2 is the azimuth angle of the observation direction,

$$R(\vartheta_0, \vartheta, \varphi) = \frac{I^{\uparrow}(\vartheta_0, \vartheta, \varphi)}{I^*(\vartheta_0)},$$
$$T(\vartheta_0, \vartheta, \varphi) = \frac{I^{\downarrow}(\vartheta_0, \vartheta, \varphi)}{I^*(\vartheta_0)} \quad (3)$$

and $I^*(\vartheta_0) = F_0\mu_0$, $\mu_0 = \cos\vartheta_0$. Here πF_0 is the hemispherical incident flux. $I^{\downarrow}$ ($I^{\uparrow}$) is the intensity of the

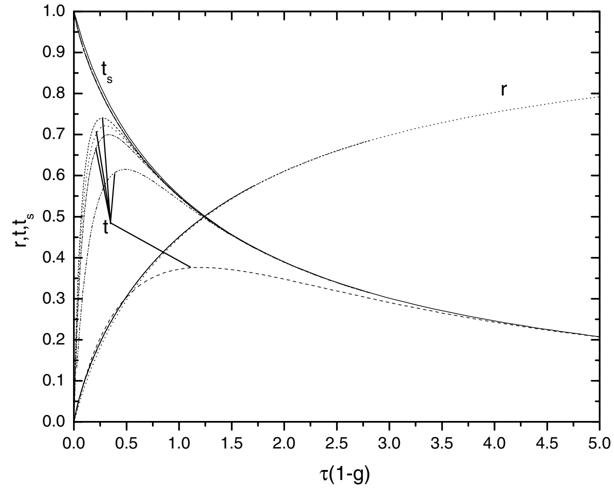

Figure 1. The dependence of the diffuse reflectance r, the diffuse transmittance t and the total transmittance t_s on $\tau(1-g)$. The results shown have been obtained by numerical solution of the integro-differential radiative transfer equation. Curves with the smaller maximum values of t correspond to the smaller values of g (see table 1). Curves $r(\tau^*)$ and $t_s(\tau^*)$ almost coincide for all cases given in table 1 (except at $g = 0.1268$, solid curve).

transmitted (reflected) diffused light, which can be obtained from the solution of the integro-differential radiative transfer equation.

The results of calculations of values r, t, and additionally the total transmittance $t_s = t + t_{dir}$, where the diffused transmittance of the direct light t_{dir} is given by

$$t_{dir} = 2\int_0^1 \mathrm{d}\mu_0\,\mu_0 \exp(-\tau/\mu_0), \tag{4}$$

are presented in figure 1. They were obtained from the numerical solution of the radiative transfer equation [1] in the framework of the discrete ordinate method [5].

Calculations were performed for a plane-parallel layer of nonabsorbing water droplets in air. It is assumed that the droplet size distribution $f(a)$ (a is the radius) is given by the following equation:

$$f(a) = \Lambda a^6 \exp(-1.5a/a_{ef}), \tag{5}$$

where Λ is the normalization constant ($\int_0^\infty f(a)\,\mathrm{d}a = 1$). The effective radius a_{ef} is defined as

$$a_{ef} = \frac{\int_0^\infty a^3 f(a)\,\mathrm{d}a}{\int_0^\infty a^2 f(a)\,\mathrm{d}a}. \tag{6}$$

The distribution (5) at the effective radius $a_{ef} = 6\ \mu$m was introduced by Deirmendjian [6] for the characterization of droplet distributions in terrestrial clouds.

The value of a_{ef} was not constant in our calculations. It was varied in the range 0.06–16.0 μm. Effective radii a_{ef} used in calculations are presented in table 1 together with asymmetry parameters:

$$g = \tfrac{1}{2}\int_{-1}^{1} p(\theta)\sin\theta\cos\theta\,\mathrm{d}\theta, \tag{7}$$

Table 1. The values of a_{ef} and g used in calculations.

a_{ef} (μ m)	g
0.06	0.1268
0.23	0.7187
0.6	0.8289
6.0	0.8499
16.0	0.8691

Table 2. Parameters in equation (10).

n	α_n	β_n
1	0.059	7.19
2	0.356	2.24
3	0.584	1.17

which were calculated using the Mie theory [6] for spherical polydispersions of water droplets. The wavelength was equal to 650 nm. Note that θ in equation (7) is the scattering angle and $p(\theta)$ is the phase function, which gives a conditional probability for photons to be scattered in a direction specified by the angle θ.

The value t_{dir} gives the diffused transmittance of the direct light and t_s is the total transmittance. Thus, due to the energy conservation law for non-absorbing layers, it follows that

$$r + t + t_{dir} = 1. \tag{8}$$

We see that data presented in figure 1 satisfy this equation. We also note that the value of r is the least sensitive to the values of a_{ef} (and thus also g), if plotted against the transport optical thickness $\tau^* = \tau(1-g)$. The value of t is most sensitive to the value of g, if plotted against τ^*.

It is known from the general theory [1] that the value of t depends almost exclusively on τ^* as τ^* becomes larger than a fixed number τ_ℓ^*, which is a function of g. This also follows from figure 1. We see that this asymptotic limit is reached for smaller values of τ_ℓ^*, if g is larger. In particular, it follows from figure 1 that τ_ℓ^* is in the range 0.5–1.5 at $g \approx 0.72$–0.87 and it is around 5 at $g = 0.13$. On the other hand, t_s is only weakly sensitive to g as plotted against τ^*.

The most convenient function for the approximation is the diffuse transmittance r (see figure 1). If it is known, then we have from equation (8)

$$t = 1 - r - t_{dir}. \tag{9}$$

The value t_{dir} can be computed from equation (4) or by using the approximation

$$t_{dir} = \sum_{n=1}^{3} \alpha_n \mathrm{e}^{-\beta_n \tau}, \tag{10}$$

where α_n, β_n at $n = 1, 2, 3$ are given in table 2. The accuracy of this equation as compared to integral (4) is better than 1% at $\tau \leqslant 4$. For larger τ, we have $t_{dir} < 0.0055$ and, therefore, the direct light almost does not contribute to t_s.

It is known [1, 7] that $r \approx r_a$, where

$$r_a = \frac{\sigma - 1 + 0.75\tau^*}{\sigma + 0.75\tau^*}, \tag{11}$$

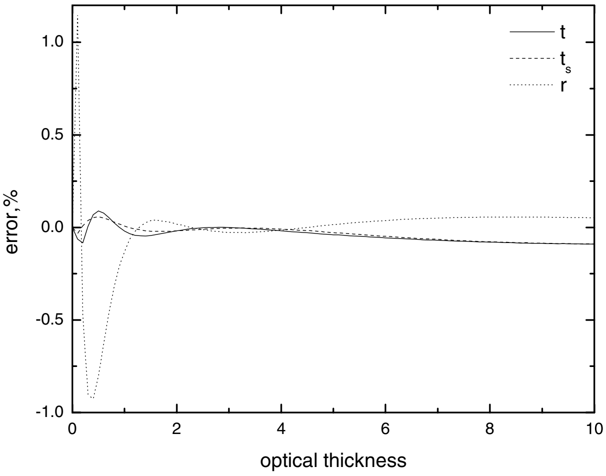

Figure 2. The errors of equation (17) with account of equations (16), (10) and (11) for values of t, t_s and r at $a_{ef} = 6\ \mu$m.

as $\tau^*/\tau_l^* > 1$. The parameter

$$\sigma = 3\int_0^1 \mathrm{d}\mu\, \mu^2 K_0(\mu) \tag{12}$$

is almost insensitive to the medium microstructure and is equal approximately [1, 7] to 1.07. Here $\mu = \cos\vartheta$ and $K_0(\mu)$ is the escape function, which is defined via the reflection function of the semi-infinite nonabsorbing turbid media $R_\infty(\mu, \mu_0, \varphi)$ by the following equation [7]:

$$K_0(\mu) = 0.75\left[\mu + 2\int_0^1 \mathrm{d}\mu_0\, \mu_0^2 R_\infty(\mu, \mu_0)\right], \tag{13}$$

where

$$R_\infty(\mu, \mu_0) = \frac{1}{2\pi}\int_0^{2\pi} R_\infty(\mu, \mu_0, \varphi)\, \mathrm{d}\varphi. \tag{14}$$

We see that $r_a(\tau = 0) = 1 - \sigma^{-1} > 0$. It follows both from numerical calculations and the physics of the problem, however, that $r(\tau = 0) \equiv 0$ (see figure 1). Thus, the difference

$$s = r_a - r \tag{15}$$

is a positive number. This is confirmed by numerical calculations as well. Our task is to approximate the function $s(\tau^*)$. Then the value of r can be found from equation (15) as $r = r_a - s$ (see also using equation (11)).

Clearly, we have the following limits for the function $s(\tau^*)$: $s(\tau^*) \to 0$ as $\tau^* \to \infty$ (because the value of r_a is the asymptotic result for r at large thickness, see equation (11)) and $s(\tau^*) \to 1 - \sigma^{-1}$ as $\tau^* \to 0$ (because then the value of $r \to 0$, and thus $s \to r_a$, for small optical thickness). These conditions are satisfied, e.g., by the expression

$$s(\tau^*) = A\exp(-B\tau^*) + C\exp(-D\tau^*), \tag{16}$$

which is similar to equation (10), but with other constants involved. In particular, we have $C = 1-\sigma^{-1}-A$. This insures that $s(0) = 1 - \sigma^{-1}$ and, correspondingly, $r(0)$, calculated from equations (15), (16) and (11), is also zero. Constants $A = 0.047$, $B = 5.26$ and $D = 25.0$ were found from the fitting procedure for the case $a_{ef} = 6\ \mu$m (see table 1).

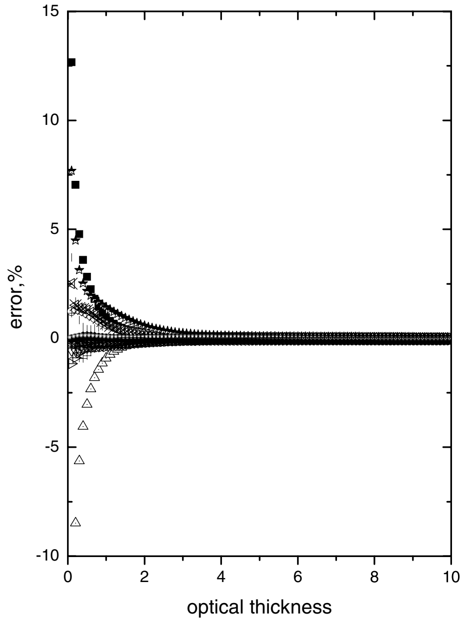

Figure 3. The errors of equations (17) with account of equations (16), (10) and (11) for values of t, t_s and r at various a_{ef}, given in table 1. Triangles with central points correspond to the smallest particles (0.06 μm) and filled squares correspond to the largest particles (16 μm). Results for other sizes of particles are distributed between these two limiting cases.

Final approximate expressions have the following forms:

$$r = r_a - s, \qquad t_s = 1 - r, \qquad t = 1 - r - t_{dir}, \tag{17}$$

where r_a, s, t_{dir} are given by equations (11), (16) and (10) respectively. We also can define $t_a = 1 - r_a$. Then it follows (see equation (11)) that

$$t_a = \frac{1}{\sigma + \frac{3}{4}\tau^*}. \tag{18}$$

This is the well known result [1, 7] for the asymptotic diffused transmittance, which is valid as $\tau^* \to \infty$.

Errors of equation (17) at $a_{ef} = 6\ \mu$m are given in figure 2. They are smaller than 1.2% for arbitrary τ. This is due to the fact that values of constants A, B, C, D were found at this value of a_{ef}. It follows from figure 3 that errors somewhat increase for other values of a_{ef}, given in table 1. In particular, they reach 15% for the value of r at $a_{ef} = 0.06\ \mu$m (triangles with central points in figure 3) and 13% for the value of t_s at $a_{ef} = 16\ \mu$m (filled squares in figure 3) at small τ, where, however, other approximations are available [4]. The errors are smaller than 1.5% at $\tau > 1$ for all cases studied.

3. Conclusion

We propose here a simple approximation (equations (17) with account of equations (16), (10) and (11)) for the calculation of the diffuse reflectance, diffuse transmittance and total transmittance of turbid nonabsorbing plane-parallel

homogeneous slabs. The accuracy of these formulae is better than 1.5% at $\tau \geqslant 1$ for all cases studied. Thus, the accuracy is better than the accuracy of measurements involved. The errors are also on the order of those of most numerical radiative transfer equation solution results. Note that the asymptotic theory (see, e.g., equation (11)) can be applied with the same accuracy only at optical thickness larger than approximately 10 (for water clouds [8]). Our modification can be used for optical thickness at least an order of magnitude smaller than that.

Formulae presented can be used to test new numerical schemes of the integro-differential radiative transfer equation solution. They are also of help in the solution of the inverse problem (e.g., the derivation of the transport extinction coefficients τ^*/L for slabs with geometrical thickness L).

All calculations were performed for a specific particle size distribution, given by equation (5). However, it is known [9] that radiative transfer characteristics do not depend very much on details of particle size distributions $f(a)$, having the same effective radius. Therefore, our results can be also applied to other types of size distribution.

Moreover, it is known [1] that equation (11) is valid for any refractive index of the particles. Therefore, we expect that our results also can be applied to light scattering media different from clouds and fogs.

Acknowledgments

This work has been made possible by the support of the Institute of Environmental Physics and Remote Sensing (University of Bremen) and the German Space Agency.

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